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MASS EARTH MOVEMENT AND TREATMENT FEASIBILITY STUDY.(U)
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B.S.

DISTRIBUTION STATEMENT A

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# DISPOSITION FORM

For use of this form, see AR 340-15; the proponent agency is The Adjutant General's Office.

| REFERENCE OR OFFICE SYMBOL | SUBJECT |
|---|---|
| WESTR | Limitation on Loan of Technical Publications |

TO Mr. Sidney Tucker
Ch, Membrance Br
Soils and Pavements Lab

FROM Ch, Publications Dist Sec

DATE 15 May 74

CMT 1

1. The present restriction on the Contract Reports listed on inclosure to our previous correspondence is due for review of statement. On 5 May 1971 we received a call from you stating that the current restriction with AMC as controlling office should be continued for an additional three years.

2. According to AR 70-31, a copy of which is inclosed, these reports should have Statement A or Statement B stamped on cover and title page. If Statement B is to be imposed on documents, please give full statement.

HANISEE

2 Incl
1. DF dtd 4 Feb 71
2. Cg 3, AR 70-31

WESSS (15 May 74)

TO C/Publications
Distribution Section

FROM C/Membrane Branch

DATE 23 May 74

CMT 2

1. Guidance furnished us previously by the sponsor (AMCRD-TV) required us to refer all requests from private sources to the sponsor for final action. However, since 1971 when the present restrictive statement was placed on the reports listed in our Memorandum dated 9 January 1968, the sponsor's office has agreed to remove the restriction from these reports.

2. We have reviewed the contents of all reports concerned and it is our judgment, at this time, that Statement A should be stamped on the cover and title pages of these reports.

TUCKER

2 Incl
nc

DA FORM 1 FEB 63 2496 REPLACES DD FORM 96, EXISTING SUPPLIES OF WHICH WILL BE ISSUED AND USED UNTIL 1 FEB 63 UNLESS SOONER EXHAUSTED. ☆ GPO 1970 O - 399-410

DISPOSITION FORM

For use of this form, see AR 340-15; the proponent agency is The Adjutant General's Office.

| REFERENCE OR OFFICE SYMBOL | SUBJECT |
| --- | --- |
| WESTR | Restriction of Reports |

TO Mr. Sidney Tucker
Chief, Membrane Section
Soils Div

FROM Publications Distribution

DATE 4 Feb 71

CMT 1

1. The restrictive statement placed on the reports listed on attached memorandum is 3 years old. In accordance with instructions contained in AR 70-31, the restriction on these reports will automatically end after 3 years unless your office has been notified to the contrary by the sponsor. If this protective statement is considered necessary and is to be extended for an additional 3 years we can so mark our Library copies (no primary distribution lists).

2. According to AR 7-31 all publications must have one of the distribution statements listed on back of Form 1064, copy of which is attached. Please indicate which of these statements should be used for these reports if the present restriction is removed.

Laura Hansen

2 Incl
as

WESSS (4 Feb 71)

TO: Publications Distribution FROM: Chief
Membrane Section

DATE: 26 Mar 71 CMT 2

Mr. W. G. Shockley, Chief, M&E Division, has requested the sponsor (AMC) to furnish us the distribution statement desired for Form 1064. As soon as this information is received from the sponsor, it will be forwarded to you.

S. J. Tucker
TUCKER

2 Incl
nc

REPLACES DD FORM 96, EXISTING SUPPLIES OF WHICH WILL BE

DEPARTMENT OF THE ARMY
WATERWAYS EXPERIMENT STATION, CORPS OF ENGINEERS
VICKSBURG, MISSISSIPPI 39180

IN REPLY REFER TO: WESSS

9 January 1968

MEMORANDUM FOR: CHIEF, REPRODUCTION AND REPORTS BRANCH (THRU CHANNELS)
CHIEF, TECHNICAL SERVICES DIVISION

SUBJECT: Release of Contract Reports Concerning Studies in the Fields of Mass Movement, Disaggregation, and Stabilization of Soil

1. It is requested that the subject contract reports be released only for Department of Defense (DOD) use and to DOD contractors that have an established need to know. Should requests be obtained from private sources other than those mentioned above, we have been instructed by AMCRD-TV to refer these requests to them for final action.

2. Reports to be given the above-mentioned distribution are as follows:

| Number | Date | Contractor and Report Title |
|---|---|---|
| 3-92 | Nov 1964 | Wilson, Nuttall, Raimond Engineers, Inc., "Conceptual Studies in the Fields of Mass Movement of Earth and Mass Stabilization of Earth" |
| 3-137 | Mar 1964 | General American Transportation Corp., "Conceptual Studies in the Fields of Mass Movement of Earth and Stabilization of Earth" |
| 3-138 | Apr 1964 | Southwest Research Institute, "A Feasibility Study of Mass Movement, Disaggregation, and Stabilization of Soil" |
| 3-144 | May 1965 | Southwest Research Institute, "Feasibility Study on the Rapid Stabilization of Soils by the Use of Sulfur" |
| 3-149 | Feb 1966 | Southwest Research Institute, "Design, Fabrication and Development of a Single Explosion Cell for the Repetitive Explosion Device for Soil Displacement" |
| 3-139 | Apr 1964 | University of Virginia, "Conceptual Studies in the Field of Mass Movement of Earth and Stabilization of Earth" |

WESSS 9 January 1968

SUBJECT: Release of Contract Reports Concerning Studies in the Fields of Mass Movement, Disaggregation, and Stabilization of Soil

| Number | Date | Contractor and Report Title |
|---|---|---|
| 3-140 | May 1964 | Texas Instruments Inc., "An Investigation of Some Problems Concerned with Thermal Soil Stabilization Processes" |
| 3-141 | Aug 1964 | Leo Casagrande, "Conceptual Studies in the Fields of Mass Movement of Earth and Stabilization of Earth" |
| 3-142 | | Rensselaer Polytechnic Institute, "Mass Earth Movement and Treatment Feasibility Study" |
| 3-157 | Nov 1964 | Roland F. Beers, Inc., Alexandria, Virginia, "Mass Earth Movement and Treatment with Nuclear Explosives" |

(All of the above reports are listed on pages G-37 through G-39 of the List of Publications Supplement dated 1 January 1967.)

3. When additional contract reports are completed that concern mass movement, disaggregation, and stabilization of soil, we will indicate the proper distribution that should be given to these reports, as I am sure that, heretofore, distribution for the above-listed reports was not indicated properly to the Library and the Reports Section.

S. G. TUCKER
Engineer
Chief, Membrane Section

DOD-IMPOSED DISTRIBUTION STATEMENT (AR 70-31, Chg 3)

Approved for public release; Distribution unlimited.
Date: 23 May 1974

AD-A040179

WES CONTRACT REPORT 3-142

# MASS EARTH MOVEMENT AND TREATMENT FEASIBILITY STUDY

by

E.C.W.A. Geuze

Prepared For
U. S. Army Engineer Waterways Experiment Station
Corps of Engineers
Vicksburg, Mississippi

Contract No. DA-22-079-eng-386
Sponsored By U. S. Army Materiel Command
Project No. 1-T-0-21701-A-046



The Winslow Laboratories
Rensselaer Polytechnic Institute
Troy, New York

# MASS EARTH MOVEMENT AND TREATMENT FEASIBILITY STUDY

by

Emmericus C.W.A. Geuze
Professor of Soil Mechanics and Foundation Engineering
Rensselaer Polytechnic Institute

## SYNOPSIS

The basic concept of desaggregation is to develop a state of failure in soil media by a centre of pressure located at a convenient depth below the surface of the soil mass.

The mechanism of failure development in the medium located between the centre of pressure and the soil surface has been based upon the results obtained by application of the theory of plasticity of granular and cohesive materials.

The failure properties of these two soil groups have been expressed by the parameters of internal friction and of cohesion.

The basic problem is presented in an analytical form which yields approximate solutions for the pressures required to cause failure in the medium at different depths of the centre of pressure and with different diameters of the pressure cavity.

The consequences of the approximations are discussed and an altogether new theory is developed. The results of an application of this theory to the failure of a granular material are shown and a comparison with results obtained by the approximate analytical solution is made.

Liquid loading characteristics are discussed in relation to the compressibility and permeability of the soil materials.

A model experiment has been used to demonstrate the development of the failure area and to provide pressure data.

Analysis of the basic problem

The pressure cavity is supposed to have the shape of a sphere with radius $r_o$. Its centre be located at the depth D below the horizontal boundary of the soil medium. Pressure is applied to the wall of the cavity through the intermediary of a thin rubber membrane. The cavity is completely filled with liquid (water), which is held under pressure (Fig. 1).

A certain pressure is required to maintain the initial diameter of the cavity. An increase of pressure is accompanied by outward radial compression of the soil material around the cavity. The compressive strains decrease rather rapidly with increasing distance from the centre of the cavity; approximately proportional to the one-third power of that distance.

In a dry, or moist, granular material compressions and the subsequent displacement of the cavity wall occur instantaneously. Compaction in the immediate vicinity of the cavity is accompanied by the formation of a dense shell of material. When the material is saturated with water, the rate of compression depends on the compressibility and permeability characteristics of the material and on the duration of the imposed loading increment.

Though the concept "rate of consolidation" usually is not applied to granular materials, the process of radial compression in the saturated medium described previously does not differ essentially from that in a saturated cohesive material.

A rapid build-up of the cavity pressure holds the depth of penetration of the consolidation effect to a minimum. Failure of the material occurs when a certain critical value of the cavity pressure is attained. The failure wave propagates through the material in upward direction as long as the pressure is maintained at

the critical value until it reaches the upper boundary of the medium. When the cavity pressure is kept at the same value beyond this point, the body of soil within the failure area continues to move in upward direction.

When the impervious membrane is left out water will penetrate into the medium as a result of the increase of cavity pressure. In a dry medium the penetration of water, along with the displacement of air in the voids, occurs as a capillary saturation. The rate of penetration depends on the compressibility, the permeability and the capillary suction force of the soil. In a saturated medium the rate of penetration depends on compressibility and permeability only.

The mechanism of this penetration process has identical consequences for the transfer of the liquid load to the soil medium as when an impervious membrane would separate the soil from the liquid, i.e., the load acts upon the medium through the intermediary of a spherical soil shell. This shell is composed of more or less compacted soil. Its thickness depends on the rate of building up of the cavity pressure and on the soil characteristics as mentioned earlier.

## Chapter I

### The approximate analytical solution of the basic problem

An approximate analytical solution has been given by NADAI [1] for the problem of a small spherical cavity held under static pressure, situated deep under the surface of granular material.

NADAI solved the problem by making two simplifying assumptions:

a. The state of stress in the vicinity of the axis OA (Fig. 2) is taken approximately equal to that in a spherical body of the same material around O, tangent to the surface at A.

b. The gravity field for the material, which has the direction AO at every point in the medium, is replaced by a field of body forces converging radially toward the centre O and of the same magnitude as the uniform force of gravity. The approximate nature of Nadai's solution is a consequence of these assumptions.

The equation of equilibrium of an element of the medium in the shape of a truncated circular cone depends on the radius (Fig. 3) only, because of the polar symmetry of the stress field.

Equilibrium of the element in a radial direction requires that:

$$\sigma_r . r\,d\theta . r\,d\theta - (\sigma_r + d\sigma_r).(r+dr)d\theta.(r+dr)d\theta + 2\sigma_t . \sin\tfrac{1}{2}d\theta . \tfrac{1}{2}\{r.d\theta + (r+dr)d\theta\}dr$$
$$- \gamma . \tfrac{1}{2}\{r\,d\theta . r\,d\theta + (r+dr)d\theta.(r+dr)d\theta\}dr = 0$$

Dividing through by $r^2(d\theta)^2$ and neglecting $dr/r$ with respect to unity:

$$r\frac{d\sigma_r}{dr} + 2(\sigma_r - \sigma_t) + \gamma . r = 0 \qquad (1)$$

This equation can be solved under the assumption that failure occurs simultaneously along sliplines located in radial planes through the axis OZ and in tangential planes through O at right angles to the radial planes. The major principal stress $\sigma_r$ acts in radial direction, both minor principal stresses $\sigma_t$ in tangential directions.

A. Granular material

We introduce the Mohr-Coulomb condition of failure in the equation of equilibrium:

$$\sigma_t = \frac{1-\sin\varphi}{1+\sin\varphi} . \sigma_r \qquad (2)$$

Putting: $\frac{4\sin\varphi}{1+\sin\varphi} = a$ (a parameter of the solution)

Eq. (1) takes the form:

$$r\frac{d\sigma_r}{dr} + a.\sigma_r + \gamma . r = 0 \qquad (3)$$

The solution of this equation proceeds by introducing a new variable:

$$\chi = \frac{\sigma_r}{r} \tag{4}$$

hence

$$\frac{d\sigma_r}{dr} = \chi + r\cdot\frac{d\chi}{dr}$$

and Eq. (3) takes the form:

$$r\left(\chi + r\frac{d\chi}{dr}\right) + a.r.\chi + \gamma.r = 0$$

after rearranging terms and dividing through by r:

$$\frac{d\chi}{[(a+1)\chi + \gamma]} = -\frac{dr}{r}$$

and as:

$$d\chi = \frac{1}{(a+1)}\, d[(a+1)\chi + \gamma]$$

$$\frac{1}{(a+1)}\,\frac{d[(a+1)\chi+\gamma]}{[(a+1)\chi+\gamma]} = -\frac{dr}{r}$$

Assuming failure to occur at a cavity pressure $p_o$, we have the boundary condition:

$$\sigma_r = p_o\ ; \quad \chi_o = \frac{p_o}{r_o}\ ; \quad r = r_o$$

hence:

$$\frac{1}{(a+1)}\int_{\chi=\chi_o}^{\chi=\chi}\frac{d[(a+1)\chi+\gamma]}{[(a+1)\chi+\gamma]} = -\int_{r=r_o}^{r=r}\frac{dr}{r}$$

yielding the equation

$$\ln\frac{[(a+1)\chi+\gamma]}{[(a+1)\chi_o+\gamma]} = \ln\left(\frac{r_o}{r}\right)^{(a+1)}$$

from which we obtain:

$$[(a+1)\chi+\gamma] = [(a+1)\chi_o+\gamma]\left(\frac{r_o}{r}\right)^{(a+1)}$$

and after replacing $\chi$ by $\sigma_r/r$ and $\chi_o$ by $p_o/r_o$ :

$$\left(\frac{\sigma_r}{r}\right) = \left[\left(\frac{p_o}{r_o}\right) + \frac{\gamma}{(a+1)}\right]\left(\frac{r_o}{r}\right)^{(a+1)} - \frac{\gamma}{(a+1)}$$

Multiplying through by r yields:

$$\sigma_r = \left[p_o + \frac{\gamma r_o}{(a+1)}\right]\left(\frac{r_o}{r}\right)^{a} - \frac{\gamma.r}{(a+1)} \tag{5}$$

The validity of this equation is limited by the condition that the state failure can only exist when the major principal stress $\sigma_r \geqq 0$ , because the Mohr-Coulomb failure condition is expressed by:

$$\sigma_r = \left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)\cdot\sigma_t$$

according to Eq. (2).

Hence the external boundary of the spherical medium, when free of stresses, represents the limit of the medium in failure.

This condition provides the other boundary condition required for determining the radius of the spherical body of failure.

It follows that in order to find the magnitude of this radius, we put $\sigma_r = 0$ for a radius $r_1$ in Eq. (3).

This substitution yields:

$$\left[p_o + \frac{\gamma.r_o}{(a+1)}\right]\left(\frac{r_o}{r_1}\right)^a = \frac{\gamma.r_1}{(a+1)}$$

from which we can derive:

$$\left(\frac{r_1}{r_o}\right)^{(a+1)} = (a+1)\left(\frac{p_o}{\gamma.r_o}\right) + 1 \qquad (6)$$

The ratio $r_1/r_0$ therefore depends on the absolute value of $r_0$.

Radius of failure

The radius of failure can be computed from:

$$r_1 = r_o\left[(a+1)\cdot\left(\frac{p_o}{\gamma.r_o}\right) + 1\right]^{\frac{1}{(a+1)}} \qquad (7)$$

when the quantities $a$, $\gamma$, $p_o$ and $r_o$ are given.

If the quantity $r_1$ is given besides the quantities $a$, $\gamma$ and $r_0$, the cavity pressure $p_o$ can be computed from:

$$p_o = \frac{\gamma \cdot r_o}{(a+1)}\left[\left(\frac{r_1}{r_o}\right)^{(a+1)} - 1\right] \tag{8}$$

Fig. 4 and 5 show diagrams of $p_o$-$r_1$ relationships for $r_0 = 1, 5, 10$ and 15 cm. respectively. The volume weight of the material $\gamma$ was taken as 1 gram/cm$^3$ and the angle of internal friction of the granular material $\varphi = 30^o$. Hence the coefficient $a = 4/3$.

It should be noted that the radius of the cavity has a considerable effect on the magnitude of $p_o$ since it appears in the denominator of the ratio $r_1/r_o$. Granular materials are characterized by an angle of internal friction $\varphi > 0$. Hence the factor $(a + 1)$ has values larger than unity.

Model rules

Eq. (8) is of significant value in establishing the design of model experiments as will be shown later.

Since both the volume weight $\gamma$ and the factor $(a + 1)$ can be adjusted in the experiment according to prototype conditions, we can establish the following model relationships:

$$P_o = \frac{\gamma \cdot R_o}{(a+1)}\left[\left(\frac{R_1}{R_o}\right)^{(a+1)} - 1\right]$$

$$p_o = \frac{\gamma \cdot r_o}{(a+1)}\left[\left(\frac{r_1}{r_o}\right)^{(a+1)} - 1\right]$$

where capitals indicate the relationship for the prototype and the small characters indicate the relationship for experimental conditions.

Setting:

$$\frac{R_1}{R_o} = \frac{r_1}{r_o} = n \tag{9}$$

we obtain the ratio:

$$\frac{P_o}{p_o} = \frac{R_o}{r_o}$$

or:

$$P_o = \frac{R_o}{r_o} p_o \tag{10}$$

Eqs. (9) and (10) establish the model rules for experiments on a small scale, which can be used to predict prototype behavior.

Rate of dissipation of the cavity pressure within the medium.

The rate of dissipation of the cavity pressure is expressed by Eq. (5). When dividing through by $\gamma . r$ the following equation results:

$$\left(\frac{\sigma_r}{\gamma . r}\right) = \left[\left(\frac{p_o}{\gamma . r_o}\right) + \frac{1}{(a+1)}\right] \left(\frac{r_o}{r}\right)^{(a+1)} - \frac{1}{(a+1)} \tag{11}$$

When the quantities $\gamma$, a, $r_0$ and $p_o$ are given constants Eq. (11) can be written in a simplified form:

$$\left(\frac{\sigma_r}{\gamma . r}\right) = (A+B) \cdot \left(\frac{r_o}{r}\right)^{1/B} - B$$

The rate of dissipation depends mostly on the magnitude of the power 1/B, i.e., on the angle of internal friction $\varphi$. As a demonstration of the rate of dissipation we take $\gamma$ = 1 gram/cm$^3$ and $\varphi = 19^o 26'$.

Then:

$$\left(\frac{\sigma_r}{r}\right) = \left(A + \frac{1}{2}\right) \cdot \left(\frac{r_o}{r}\right)^2 - \frac{1}{2}$$

i.e., the ratio of $\sigma_r / r$ is approximately proportional to the square of the ratio $r_0/r$.

Fig. 6 shows a diagram of the $\sigma_r$-r relationships when $\gamma$ =1, $r_0$ = 10 cm, a=4/3 and $p_o = 10^3$, $10^4$ and $5.10^4$ grams/cm$^2$.

The slipplane field

The slip planes intersect the principal stress trajectories at angles:

$$\left.\begin{aligned} \beta_1 &= \psi + \left(\frac{90^o - \varphi}{2}\right) = \psi + \delta \\ \beta_2 &= \psi - \left(\frac{90^o - \varphi}{2}\right) = \psi - \delta \end{aligned}\right\} \psi = \text{angle of principal stress direction}$$

The trajectories of major principal stress are a family of straight lines radiating from the centre of the cavity. The trajectories of minor principal stress are a family of concentric spherical surfaces.

The family of slip planes intersecting the family of radii at an angle of $(\psi + \delta)$ are called $s_1$-planes, those at an angle of $(\psi - \delta)$ are called $s_2$-planes. The $s_1$-planes and $s_2$-planes are conjugate planes.

Each set of conjugate planes forms a body of rotational symmetry with respect to a radius. The intersection of this body with a plane passing through the radius is a set of conjugate slip lines, which are also called $s_1$- and $s_2$-lines.

The derivation of the $s_1$- and $s_2$-slip lines follows from the basic property of the direction of slip with respect to the direction of major principal stress in the medium.

At point P in the diagram of Fig. 7 the tangent to the $s_1$-slip line has the direction $(\psi + \delta)$ and to the $s_2$-slip line the direction $(\psi - \delta)$ with respect to the Z-axis.

The tangent to the $s_1$-slip line satisfies:

$$\frac{\partial R}{R . \partial \psi} = \cot \delta$$

Taking a point $R_o$, $-\psi_o$ at the boundary of the cavity (Fig. 8) as the origin of the slip line:

$$\int_{R=R_o}^{R=R} \frac{\partial R}{R} = \cot \delta \int_{\psi=-\psi_o}^{\psi=\psi} \partial \psi$$

Hence:

$$\ln \frac{R}{R_o} = \cot \delta (\psi + \psi_o)$$

and:

$$R = R_o \exp \left[ \cot \delta . (\psi + \psi_o) \right] \tag{12}$$

The tangent to the $s_2$-slip line at P satisfies:

$$\frac{\partial R}{R.\partial\psi} = -\cot n\,\delta$$

because the sign of $\partial\psi$ is negative.

Taking a point $R_o, +\psi_o$ as the origin of the $s_2$-slip line:

$$\int_{R=R_o}^{R=R}\frac{\partial R}{R} = -\cot n\,\delta\int_{\psi=\psi_o}^{\psi=\psi}\partial\psi$$

Hence:

$$R = R_o.\exp\left[-\cot n\,\delta(\psi-\psi_o)\right] \tag{13}$$

At the point of intersection of the two slip lines the radii $R_1$ from (12) and (13) are the same; hence

$$R_o.\exp\left[\cot n\,\delta(\psi_1+\psi_o)\right] = R_o.\exp\left[-\cot n\,\delta(\psi_1-\psi_o)\right]$$

giving:

$$\psi_1+\psi_o = -\psi_1+\psi_o$$

and:

$$\psi_1 = 0$$

and by substitution in either Eq. (12) or Eq. (13), we obtain:

$$\frac{R_1}{R_o} = \exp\left[\cot n\,\delta.\psi_o\right] \tag{14}$$

where:

$R_1$ = radius of the failure area.

From Eq. (14) follows that the ratio between the radius of failure $R_1$ and of the cavity $R_o$ increases with the power $\cot n\,\delta.\psi_o$.

The area of the surface of a spherical segment with radius $R_o$ and with a centre angle $2\psi_o$ represents the loaded area.

The area of this surface is:

$$2\pi R_o^2(1-\cos\psi_o)$$

The pressure acting on this area can be obtained from Eq. (8) for given magnitudes of $R_o$, $R_1$ and for $\delta = 30^o$.

The diameter of the failure area is significant, because it establishes the degree of approximation involved by the assumption of a radial body force.

From Fig. 8 it can be seen that the strongest deviation of the radial direction from the axis of symmetry (i.e., the direction of the gravity force) occurs at a point where the tangent to either the $s_1$ - or $s_2$ -slip line has the direction of the Z-axis.

At this point:

$$\psi - \delta = 0$$

and

$$-\psi + \delta = 0$$

Hence:

$$\psi_t = \delta$$

From Eq. (12) follows by substitution:

$$R_t = R_o \cdot exp\left[cotn\,\delta . (\delta + \psi_o)\right]$$

where: $R_t$ = the radius from the origin to the tangent point on the slip line.

These tangent points are located on a circle with radius $r_t$, where:

$$r_t = R_t . sin\,\delta$$

giving:

$$r_t = R_o\,sin\,\delta . exp\left[cotn\,\delta . (\delta + \psi_o)\right]$$

and using Eq. (14) for the expression of $R_1$:

$$R_o = R_1 . exp\left[-cotn\,\delta . \psi_o\right]$$

we obtain by substitution: $P$

$$r_t = R_1\,sin\,\delta . exp\left[\delta . cotn\,\delta\right]$$

and:

$$r_t / R_1 = sin\,\delta . exp\left[\delta . cotn\,\delta\right] \qquad (15)$$

For a given granular material, $\delta$ is a constant. The maximum diameter of the failure area is therefore proportional to the depth of the cavity. The radial direction of the body forces at the level of maximum diameter is independent of the depth of the cavity, because:

$$r_t / R_t = sin\,\delta$$

NADAI's approximate solution is therefore subject to a serious deviation from actual prototype conditions, where the direction of the body force at the level of maximum diameter may differ from the direction of gravity as much as by an angle $\delta = (45^{o} - \frac{\varphi}{2})$. This deviation decreases with increasing magnitude of $\varphi$, i.e., of the angle of internal friction of the material.

B. Cohesive material

In a cohesive medium with no internal friction, $\varphi = o$, which leaves us with only one parameter for the failure condition of the material, i.e., the cohesion c.

The minor principal stress $\sigma_t$ then obtains the value:

$$\sigma_t = \sigma_z - 2c$$

Substitution of this expression in the equation of radial equilibrium (1) yields:

$$z.\frac{\partial \sigma_z}{\partial z} + z.\gamma + 4c = 0 \tag{16}$$

and:

$$d\sigma_z = -\gamma.dz - 4c\frac{dz}{z}$$

With $\sigma_z = p_o$ at $z = z_o$:

$$\int_{\sigma_z = p_o}^{\sigma_z = \sigma_z} d\sigma_z = -\gamma\int_{z=z_o}^{z=z} dz - 4c\int_{z=z_o}^{z=z}\frac{dz}{z}$$

hence:

$$\sigma_z = p_o - \gamma(z - z_o) - 4c.\ln\left(\frac{z}{z_o}\right) \tag{17}$$

At $r = r_1$, $\sigma_r = o$, yielding:

$$p_o = \gamma(z_1 - z_o) + 4c.\ln\left(\frac{z_1}{z_o}\right) \tag{18}$$

In dimensionless quantities:

$$\left(\frac{p_o}{\gamma.z_o}\right) = \left(\frac{z_1}{z_o}\right) - 1 + \left(\frac{4c}{\gamma.z_o}\right).\ln\left(\frac{z_1}{z_o}\right)$$

The slip line field

The equations of the slip lines are directly obtained from Eqs. (12) and (13), by putting $\varphi = o$, giving:

$$\delta = \left(\frac{90^\circ - \varphi}{2}\right) = 45^\circ$$

hence: $\cot n\,\delta = 1$

and the Eqs. change to:

$$R = R_o.\exp(\psi + \psi_o) \tag{19}$$

and

$$R = R_o.\exp(-\psi + \psi_o) \tag{20}$$

for $s_1$- lines and $s_2$-lines respectively.

The point of intersection follows from the equality of $R_1$ at $\psi_1$:

$$\psi_1 = 0$$

and by substitution we obtain:

$$\frac{R_1}{R_0} = \exp(\psi_0) \tag{21}$$

Other relationships as previously obtained:

$$\psi_t = 0$$

$$R_t = R_0 \cdot \exp(\psi_0 + 45°)$$

$$\tau_t = R_t \frac{\sqrt{2}}{2}$$

and

$$\tau_t = \frac{\sqrt{2}}{2} R_0 \cdot \exp(\psi_0 + 45°)$$

and:

$$\tau_r / R_1 = \frac{\sqrt{2}}{2} \cdot \exp\left(\frac{\pi}{4}\right)$$

The same objections as previously raised apply to a cohesive material. The direction of the radial body force deviates by as much as 45° from the actual direction of gravity at the level of maximum diameter of the failure area.

## Conclusions

The analysis of an approximate solution following Nadai's approach yields results which may show reasonably correct values in the vicinity of the axis of symmetry with a direction parallel to the force of gravity.

The slip line field extends to a considerable distance from the axis of symmetry for both granular and cohesive materials. At this distance the direction of a body force in the radial direction, as assumed in the approximate solution, shows a considerable deviation from the direction of the force of gravity.

A solution as suggested by Nadai cannot be taken as a basis for the convenient computation of the quantities required before it proves its validity in comparison with the results of an exact solution of the problem.

Chapter II

A. The failure of the soil by cavity pressure, when the medium is subjected to a uniform field of gravity forces.

The basic assumptions are identical with those made in the derivations of the preceding analysis.

The stress field is symmetrical with respect to the direction of the gravity force. The vertical axis is taken opposite to that direction.

Using cylindrical coordinates, the equilibrium equations can be derived from the equilibrium of a cylindrical element (Fig. 9). Since the stress field is symmetrical to OZ, radial planes are subjected to a normal stress only. Shear stresses act on the planes r = constant and z = constant.

We have for equilibrium of forces in Z-direction:

$$\left[\sigma_z - \left(\sigma_z + \frac{\partial \sigma_z}{\partial z}dz\right)\right]\cdot\frac{1}{2}(2r+dr).d\theta.dr + \tau.r\,d\theta.dz - \left(\tau+\frac{\partial \tau}{\partial r}.dr\right)(r+dr)\,d\theta.dz - \gamma.\frac{1}{2}(2r+dr)\,d\theta.dr.dz = 0$$

$$-\frac{\partial \sigma_z}{\partial z}\cdot\frac{1}{2}(2r+dr).d\theta.dr.dz - \tau.dr.d\theta.dz - r.\frac{\partial \tau}{\partial r}.dr.d\theta.dz - \frac{\partial \tau}{\partial r}.dr.dr.d\theta.dz - \gamma.\frac{1}{2}(2r+dr)\,d\theta.dr.dz = 0$$

Dividing through by $r.d\theta.dr.dz$ gives:

$$-\frac{\partial \sigma_z}{\partial z}\left(1+\frac{dr}{r}\right) - \frac{\tau}{r} - \frac{\partial \tau}{\partial r}\left(1+\frac{dr}{r}\right) - \gamma\left(1+\frac{dr}{r}\right) = 0$$

Since we may neglect $dr/r$ with respect to unity:

$$\frac{\partial \sigma_z}{\partial z} + \frac{\tau}{r} + \frac{\partial \tau}{\partial r} + \gamma = 0 \qquad (22)$$

Equilibrium of forces in r-direction:

$$\sigma_r . r d\theta . dz - \left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right)(r + dr) d\theta . dz + 2\sigma_t . \sin\tfrac{1}{2} d\theta . dr . dz +$$
$$\left[\tau - \left(\tau + \frac{\partial \tau}{\partial z} dz\right)\right] . \frac{1}{2}(2r + dr) . d\theta . dr = 0$$

and with: $\sin\frac{1}{2} d\theta = \frac{1}{2} d\theta$

$$-\frac{\partial \sigma_r}{\partial r} . r . d\theta . dr . dz - \sigma_r . d\theta . dr . dz - \frac{\partial \sigma_r}{\partial r} dr . d\theta . dr . dz + \sigma_t . d\theta . dr . dz$$
$$-\frac{\partial \tau}{\partial z}\left(r + \tfrac{1}{2} dr\right) d\theta . dr . dz = 0$$

Dividing through by $r . d\theta . dr . dz$ yields:

$$-\frac{\partial \sigma_r}{\partial r} - \frac{\sigma_r}{r} - \frac{\partial \sigma_r}{\partial r} \cdot \frac{dr}{r} + \frac{\sigma_t}{r} - \frac{\partial \tau}{\partial z}\left(1 + \frac{1}{2} \cdot \frac{dr}{r}\right) = 0$$

and:

$$\frac{\partial \sigma_r}{\partial r}\left(1 + \frac{dr}{r}\right) + \frac{\sigma_r}{r} - \frac{\sigma_t}{r} + \frac{\partial \tau}{\partial z}\left(1 + \frac{1}{2} \cdot \frac{dr}{r}\right) = 0$$

Neglecting $\frac{dr}{r}$ with respect to unity:

$$\frac{\partial \sigma_r}{\partial r} + \frac{(\sigma_r - \sigma_t)}{r} + \frac{\partial \tau}{\partial z} = 0 \qquad (23)$$

Eqs. (22) and (23) can be simplified by the following substitutions:

$$r . \frac{\partial \sigma_z}{\partial z} = \frac{\partial}{\partial z}(r . \sigma_z) \quad ; \quad r \frac{\partial \sigma_r}{\partial r} + \sigma_r = \frac{\partial}{\partial r}(r . \sigma_r)$$

$$r . \frac{\partial \tau}{\partial z} = \frac{\partial}{\partial z}(r . \tau) \quad ; \quad r . \frac{\partial \tau}{\partial r} + \tau = \frac{\partial}{\partial r}(r . \tau)$$

yielding:

$$\frac{\partial}{\partial r}(r . \tau) + \frac{\partial}{\partial z}(r . \sigma_z) = -\gamma . r \qquad (24)$$

$$\frac{\partial}{\partial r}(r . \sigma_r) + \frac{\partial}{\partial z}(r . \tau) = \sigma_t \qquad (25)$$

The Mohr-Coulomb concept of failure requires tangency of the stress circles to the failure envelope (Fig. 10).

The relation between the stresses then is:

$$\frac{1}{2}(\sigma_r + \sigma_z) \cdot \sin\varphi = \left[\left\{\frac{1}{2}(\sigma_r - \sigma_z)\right\}^2 + \tau^2\right]^{\frac{1}{2}} \qquad (26)$$

Introduction of the failure condition (26) into the differential equations of equilibrium are simplified by following Nadai's approach [2] .

The stress $\sigma_r$, $\sigma_z$ and $\tau$ are expressed in terms of the variables $\psi$ and $\sigma_m$ :

$\psi$ = angle of the r-axis with the positive direction of the major principal stress

$\sigma_m = 1/2\ (\sigma_r + \sigma_z)$ = mean value of the radial and axial normal stresses.

From Fig. 10 follows that:

$$\begin{aligned} \sigma_r &= \sigma_m (1 + \sin\varphi . \cos 2\psi) \\ \sigma_z &= \sigma_m (1 - \sin\varphi . \cos 2\psi) \\ \tau &= \sigma_m (\quad \sin\varphi . \sin 2\psi) \end{aligned} \qquad (27)$$

According to the Mohr-Coulomb concept the intermediate principal stress does not affect the ratio between the major and the minor principal stress at failure. Since we assumed a failure field of rotational symmetry with respect to the z-axis, the intermediate principal stress is the tangential normal stress $\sigma_t$ acting on radial planes through OZ. The tangential normal stress can therefore obtain any value between the principal stresses $\sigma_1$ and $\sigma_3$ without infringing the failure conditions. Since the mean value of $\sigma_1$ and $\sigma_3$ equals the mean value of $\sigma_r$ and $\sigma_z$ :

$$\sigma_m = \frac{1}{2}(\sigma_1 + \sigma_3) = \frac{1}{2}(\sigma_r + \sigma_z)$$

The magnitudes of $\sigma_1$ and $\sigma_3$ can be obtained from the first and second Eq. of (27) by taking $\psi = 0$:

$$\sigma_1 = (1+\sin\varphi)\,\sigma_m$$
$$\sigma_3 = (1-\sin\varphi)\,\sigma_m$$

The tangential stress $\sigma_t$ therefore satisfies:

$$\sigma_1 > \sigma_2 = \sigma_t > \sigma_3$$

and we can write:

$$\sigma_t = (1+\alpha.\sin\varphi).\sigma_m \tag{28}$$

where: $-1 < \alpha < +1$

Elimination of $\sigma_r$, $\sigma_z$ and $\tau$ from Eqs. (24) and (25) with the help of Eqs. (27) proceeds as follows:

$$r.\sigma_r = (r.\sigma_m)(1+\sin\varphi.\cos 2\psi)$$
$$r.\sigma_z = (r.\sigma_m)(1-\sin\varphi.\cos 2\psi)$$
$$r.\tau = (r.\sigma_m).\sin\varphi.\sin 2\psi$$

from which we obtain the following derivatives:

$$\frac{\partial}{\partial r}(r.\tau) = \sin\varphi.\sin 2\psi.\frac{\partial}{\partial r}(r.\sigma_m) + 2(r.\sigma_m).\sin\varphi.\cos 2\psi.\frac{\partial\psi}{\partial r}$$

$$\frac{\partial}{\partial z}(r.z) = (1-\sin\varphi.\cos 2\psi).\frac{\partial}{\partial z}(r.\sigma_m) + 2(r.\sigma_m).\sin\varphi.\sin 2\psi.\frac{\partial\psi}{\partial z}$$

$$\frac{\partial}{\partial r}(r.\sigma_r) = (1+\sin\varphi.\cos 2\psi).\frac{\partial}{\partial r}(r.\sigma_m) - 2(r.\sigma_m).\sin\varphi.\sin 2\psi.\frac{\partial\psi}{\partial r}$$

$$\frac{\partial}{\partial z}(r.\tau) = \sin\varphi.\sin 2\psi.\frac{\partial}{\partial z}(r.\sigma_m) + 2(r.\sigma_m).\sin\varphi.\cos 2\psi.\frac{\partial\psi}{\partial z}$$

Introduction of these expressions in Eqs. (24) and (25) yields:

$$\sin\varphi.\sin 2\psi.\frac{\partial}{\partial r}(r.\sigma_m) + 2(r.\sigma_m).\sin\varphi.\cos 2\psi.\frac{\partial\psi}{\partial r}$$
$$+ (1-\sin\varphi.\cos 2\psi).\frac{\partial}{\partial z}(r.\sigma_m) + 2(r.\sigma_m).\sin\varphi.\sin 2\psi.\frac{\partial\psi}{\partial z} = -\gamma.r \tag{29}$$

$$(1+\sin\varphi\cdot\cos 2\psi)\cdot\frac{\partial}{\partial r}(r\cdot\sigma_m) - 2(r\cdot\sigma_m)\cdot\sin\varphi\cdot\sin 2\psi\cdot\frac{\partial\psi}{\partial r}$$

$$+\sin\varphi\cdot\sin 2\psi\cdot\frac{\partial}{\partial z}(r\cdot\sigma_m) + 2(r\cdot\sigma_m)\cdot\sin\varphi\cdot\cos 2\psi\cdot\frac{\partial\psi}{\partial z} = \sigma_t \qquad (30)$$

These equations are simplified further by introduction of the slip lines as a reference system instead of the $r$-$z$ coordinates.

When the coordinates of the diagram of stresses are oriented with respect to the physical plane in such a manner that the positive direction of the $\sigma$- axis coincides with the positive direction of the r-axis (Fig. 10), the direction of the major principal stress at a point A in the physical plane is identical with the vector $Q_oP$ in the stress diagram. The slip line directions follow from the location of the tangent points $Q_1$ and $Q_2$, representing the shearing stresses at the slip planes in A. The lines $Q_1P$ and $Q_2P$ represent the shearing directions with respect to the direction of major principal stress $Q_oP$, hence:

$$\text{angle}\,Q_1PQ_o = (45^\circ - \tfrac{\varphi}{2})$$
$$\text{angle}\,Q_2PQ_o = (45^\circ - \tfrac{\varphi}{2})$$

The angle enclosed by $Q_1P$ and the positive -axis therefore is:

$$\beta_1 = \psi + (45^\circ - \tfrac{\varphi}{2})$$

and by $Q_2P$ and the positive $\sigma$-axis:

$$\beta_2 = \psi - (45^\circ - \tfrac{\varphi}{2})$$

The positive directions of the slip lines at point A in the physical plane are indicated by arrows. The slip line direction of $\beta_1$ will be called the $s_1$-direction; the direction of $\beta_2$ the $s_2$-direction.

The partial derivatives with respect to these directions can be built up according to the chain rule:

$$\frac{\partial}{\partial s_1} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial s_1} + \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial s_1}$$

$$\frac{\partial}{\partial s_2} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial s_2} + \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial s_2}$$

where:

$$\frac{\partial r}{\partial s_1} = \cos\beta_1 \quad ; \quad \frac{\partial z}{\partial s_1} = \sin\beta_1$$

$$\frac{\partial r}{\partial s_2} = \cos\beta_2 \quad ; \quad \frac{\partial z}{\partial s_2} = \sin\beta_2$$

Substitution of these expressions in the first two equations yields:

$$\frac{\partial}{\partial s_1} = \cos\beta_1\cdot\frac{\partial}{\partial r} + \sin\beta_1\cdot\frac{\partial}{\partial z} \qquad (31)$$

$$\frac{\partial}{\partial s_2} = \cos\beta_2\cdot\frac{\partial}{\partial r} + \sin\beta_2\cdot\frac{\partial}{\partial z}$$

In order to obtain Eqs. (29) and (30) in a form identical to the Eqs. (31) we perform the following operations:

Multiply Eq. (29) by $(+\cos\beta_2)$ and Eq. (30) by $(-\sin\beta_2)$ and add:

$$\frac{\partial}{\partial r}(r.\sigma_m)\left[-(1+\sin\varphi.\cos 2\psi)\sin\beta_2 + \sin\varphi.\sin 2\psi.\cos\beta_2\right]$$

$$-2(r.\sigma_m).\sin\varphi.\frac{\partial\psi}{\partial r}\left[-\sin 2\psi.\sin\beta_2 - \cos 2\psi.\cos\beta_2\right]$$

$$\frac{\partial}{\partial z}(r.\sigma_m)\left[-\sin\varphi.\sin 2\psi.\sin\beta_2 + (1-\sin\varphi.\cos 2\psi)\cos\beta_2\right]$$

$$+2(r.\sigma_m).\sin\varphi.\frac{\partial\psi}{\partial z}\left[-\cos 2\psi.\sin\beta_2 + \sin 2\psi.\cos\beta_2\right] = \qquad (32)$$

$$-\sigma_\psi.\sin\beta_2 - \gamma.r.\cos\beta_2$$

Multiply Eq. (29) by ($-\cos\beta_1$) and Eq. (30) by ($+\sin\beta_1$) and add:

$$
\begin{aligned}
&\frac{\partial}{\partial r}(r.\sigma_m)\Big[(1+\sin\varphi.\cos 2\psi).\sin\beta_1 - \sin\varphi.\sin 2\psi.\cos\beta_1\Big] \\
&-2(r.\sigma_m).\sin\varphi.\frac{\partial\psi}{\partial r}\Big[\sin 2\psi.\sin\beta_1 + \cos 2\psi.\cos\beta_1\Big] \\
&+\frac{\partial}{\partial z}(r.\sigma_m)\Big[\sin\varphi.\sin 2\psi.\sin\beta_1 - (1-\sin\varphi.\cos 2\psi)\cos\beta_1\Big] \\
&+2(r.\sigma_m).\sin\varphi.\frac{\partial\psi}{\partial z}\Big[\cos 2\psi.\sin\beta_1 - \sin 2\psi.\cos\beta_1\Big] = \\
&\qquad\qquad \sigma_t \sin\beta_1 + \gamma.r.\cos\beta_1
\end{aligned}
\tag{33}
$$

The expressions between brackets can be largely simplified by the use of:

$$\beta_1 = \psi + (45° - \varphi/2)$$

$$\beta_2 = \psi - (45° - \varphi/2)$$

giving:

$$
\begin{aligned}
\beta_1 + \beta_2 &= 2\psi \\
\beta_1 - \beta_2 &= (90° - \varphi) \\
\beta_1 &= 2\psi - \beta_2 \\
\beta_2 &= 2\psi - \beta_1
\end{aligned}
$$

The expressions between brackets are, from Eq. (32) in terms of $\beta_1$:

$$
\begin{aligned}
-(1+\sin\varphi.\cos 2\psi)\sin\beta_2 + \sin\varphi.\sin 2\psi.\cos\beta_2 &= +\cos\varphi.\cos\beta_1 \\
-\sin 2\psi.\sin\beta_2 - \cos 2\psi.\cos\beta_2 &= -\cos\beta_1 \\
+(1-\sin\varphi.\cos 2\psi).\cos\beta_2 - \sin\varphi.\sin 2\psi.\sin\beta_2 &= +\cos\varphi.\sin\beta_1 \\
+\sin 2\psi.\cos\beta_2 - \cos 2\psi.\sin\beta_2 &= +\sin\beta_1
\end{aligned}
$$

From Eq. (33) in terms of $\beta_2$:

$$
\begin{aligned}
+(1+\sin\varphi.\cos 2\psi).\sin\beta_1 - \sin\varphi.\sin 2\psi.\cos\beta_1 &= +\cos\varphi.\cos\beta_2 \\
+\sin 2\psi.\sin\beta_1 + \cos 2\psi.\cos\beta_1 &= +\cos\beta_2 \\
-(1-\sin\varphi.\cos 2\psi).\cos\beta_1 + \sin\varphi.\sin 2\psi.\sin\beta_1 &= +\cos\varphi.\sin\beta_2 \\
+\cos 2\psi.\sin\beta_1 - \sin 2\psi.\cos\beta_1 &= -\sin\beta_2
\end{aligned}
$$

Substitution of these expressions in Eq. (32) yields:

$$
\begin{aligned}
&\cos\varphi\Big[\cos\beta_1.\frac{\partial}{\partial r}(r.\sigma_m) + \sin\beta_1.\frac{\partial}{\partial z}(r.\sigma_m)\Big] + 2(r.\sigma_m)\sin\varphi\Big[\cos\beta_1.\frac{\partial\psi}{\partial r} + \sin\beta_1.\frac{\partial\psi}{\partial z}\Big] = \\
&\qquad\qquad -(\sigma_t.\sin\beta_2 + \gamma.r.\cos\beta_2)
\end{aligned}
\tag{34}
$$

and in Eq. (33):

$$\cos\varphi\left[\cos\beta_2\cdot\frac{\partial}{\partial r}(r.\sigma_m)+\sin\beta_2\cdot\frac{\partial}{\partial z}(r.\sigma_m)\right]-2(r.\sigma_m).\sin\varphi\left[\cos\beta_2\cdot\frac{\partial\psi}{\partial r}+\sin\beta_2\cdot\frac{\partial\psi}{\partial z}\right]=$$
$$+(\sigma_t.\sin\beta_1+\gamma.r.\cos\beta_1) \quad (35)$$

By comparing the expressions between brackets with those on the right hand side of Eqs. (31) it can readily be seen which substitutions can be made in Eqs. (34) and (35).

These substitutions yield:

$$\cos\varphi.\frac{\partial}{\partial s_1}(r.\sigma_m)+2(r.\sigma_m).\sin\varphi.\frac{\partial\psi}{\partial s_1}=-(\sigma_t\sin\beta_2+\gamma.r.\cos\beta_2) \quad (36)$$

$$\cos\varphi.\frac{\partial}{\partial s_2}(r.\sigma_m)-2(r.\sigma_m).\sin\varphi.\frac{\partial\psi}{\partial s_2}=+(\sigma_t.\sin\beta_1+\gamma.r.\cos\beta_1) \quad (37)$$

To obtain these equations in a form, which shows that the $s_1$-lines and $s_2$-lines are the characteristics of the solution we introduce the new variable:

$$\chi=\frac{\cos\varphi}{2\sin\varphi}.\ln(r.\sigma_m) \quad (38)$$

giving:

$$\frac{\partial\chi}{\partial s_1}=\frac{\cos\varphi}{2\sin\varphi}\cdot\frac{1}{(r.\sigma_m)}\cdot\frac{\partial}{\partial s_1}(r.\sigma_m)$$

and

$$\frac{\partial\chi}{\partial s_2}=\frac{\cos\varphi}{2\sin\varphi}\cdot\frac{1}{(r.\sigma_m)}\cdot\frac{\partial}{\partial s_2}.(r.\sigma_m)$$

and $(r.\sigma_m)=\exp(2\tan\varphi.\chi)$

Substitution into Eqs. (36) and (37) yields:

$$\frac{\partial}{\partial s_1}(\chi+\psi)=-\frac{1}{2\sin\varphi}.\exp(-2\tan\varphi.\chi)(\sigma_t.\sin\beta_2+\gamma.r.\cos\beta_2) \quad (39)$$

$$\frac{\partial}{\partial s_2}(\chi-\psi)=+\frac{1}{2\sin\varphi}.\exp(-2\tan\varphi.\chi)(\sigma_t\sin\beta_1+\gamma.r.\cos\beta_1) \quad (40)$$

If at a point A the values $\chi_A$ and $\psi_A$ of the variables $\chi$ and $\psi$ are known, Eq. (39) provides us with information about the increase of the sum $(\chi+\psi)$ along the $s_1$-slip line. The individual values of $\chi$ and $\psi$ at a point C on this slip line, however, cannot be obtained from this information only. It is therefore necessary that the values of $\chi_B$ and $\psi_B$ are known at a point B located on the $s_2$-slip line intersecting the $s_1$-slip line at C. In that case the increase of the difference $(\chi-\psi)$ provides us with the information about the increase along the $s_2$-slip line when going from B to C, using Eq. (40).

By combining the information about the increases of $(\chi+\psi)$ and $(\chi-\psi)$ along the $s_1$-slip line and the $s_2$-slip line respectively, $\chi_C$ and $\psi_C$ can be computed individually as follows:

$$(\chi_C+\psi_C) = (\chi_A+\psi_A) + d_1(\chi_A+\psi_A)$$
$$(\chi_C-\psi_C) = (\chi_B-\psi_B) + d_2(\chi_B-\psi_B)$$

where the suffixes 1 and 2 indicate the direction of the increments.

The computation of the increments requires the adjustment of the slip line elements $ds_1$ and $ds_2$ so that Eqs. (39) and (40) are satisfied.

B. Graphical finite difference

This method was developed by De Josselin de Jong [3] and applied to problems of failure in a two-dimensional medium in plane strain.

Use is made of the properties of the pole trail in the stress plane; yielding an additional set of equations..

The coordinates of the pole P are:

$$\sigma_p = (1+\sin\varphi\cdot\cos 2\psi)\cdot\sigma_m$$
$$\tau_p = \sin\varphi\cdot\sin 2\psi\cdot\sigma_m$$

Eqs. (36) and (37) require multiplication by r:

$$(r.\sigma_p) = (1 + \sin\varphi.\cos 2\psi).(r.\sigma_m)$$
$$(r.\tau_p) = \sin\varphi.\sin 2\psi.(r.\sigma_m)$$

The partial derivatives of $(r.\sigma_p)$ and $(r.\tau_p)$ with respect to $s_i$ ($i = 1, 2$) are:

$$\frac{\partial(r.\sigma_p)}{\partial s_i} = (1 + \sin\varphi.\cos 2\psi)\frac{\partial}{\partial s_i}(r.\sigma_m) - 2(r.\sigma_m).\sin\varphi.\sin 2\psi.\frac{\partial\psi}{\partial s_i} \quad (41)$$

$$\frac{\partial(r.\tau_p)}{\partial s_i} = \sin\varphi.\sin 2\psi.\frac{\partial}{\partial s_i}(r.\sigma_m) + 2(r.\sigma_m).\sin\varphi.\cos 2\psi.\frac{\partial\psi}{\partial s_i} \quad (42)$$

Multiply (41) by $\cos 2\psi$ and (42) by $\sin 2\psi$ and add:

$$2(r.\sigma_m)(\sin\varphi + \cos 2\psi) = \cos 2\psi\frac{\partial}{\partial s_i}(r.\sigma_p) + \sin 2\psi.\frac{\partial}{\partial s_i}(r.\tau_p) \quad (43)$$

Multiply (41) by $(-\sin\varphi.\sin 2\psi)$ and (42) by $(1 + \sin\varphi.\cos 2\psi)$ and add:

$$2(r.\sigma_m)(\sin\varphi + \cos 2\psi)\sin\varphi.\frac{\partial\psi}{\partial s_i} = -\sin\varphi.\sin 2\psi.\frac{\partial}{\partial s_i}(r.\sigma_p) + (1 + \sin\varphi.\cos 2\psi) \times \frac{\partial}{\partial s_i}(r.\tau_p) \quad (44)$$

By substitution of the right hand terms of:

$$\beta_1 = \psi + (45° - \varphi/2)$$
$$\beta_2 = \psi - (45° - \varphi/2)$$

we obtain:

$$(\sin\varphi + \cos 2\psi) = 2\cos\beta_1.\cos\beta_2$$

Multiplication of Eq. (43) by $\cos\varphi$ yields:

$$\cos\varphi.\frac{\partial}{\partial s_i}(r.\sigma_m) = \frac{1}{2\cos\beta_1.\cos\beta_2}\left[\cos\varphi.\cos 2\psi\frac{\partial}{\partial s_i}(r.\sigma_p) + \cos\varphi.\sin 2\psi.\frac{\partial}{\partial s_i}(r.\tau_p)\right] \quad (45)$$

and from Eq. (44) we obtain:

$$(r.\sigma_m).\sin\varphi\frac{\partial}{\partial s_i}(r.\sigma_m) = \frac{1}{2\cos\beta_1\cos\beta_2}\left[-\sin\varphi.\sin 2\psi.\frac{\partial}{\partial s_i}(r.\sigma_p) + (1 + \sin\varphi.\cos 2\psi).\frac{\partial}{\partial s_i}(r.\tau_p)\right] \quad (46)$$

By adding Eqs. (45) and (46) we obtain the left hand term of Eq. (36); by subtraction, the left hand expression of Eq. (37). Then introducing for $(2\psi + \varphi) = (2\beta_2 + 90^\circ)$ and for $(2\psi - \varphi) = (2\beta_1 - 90^\circ)$, we obtain:

$$\sin\beta_2 \cdot \frac{\partial}{\partial s_1}(r.\sigma_p) - \cos\beta_2 \cdot \frac{\partial}{\partial s_1}(r.\tau_p) = \cos\beta_1 \left(\sigma_t . \sin\beta_2 + \gamma . r . \cos\beta_2\right) \tag{47}$$

$$\sin\beta_1 \cdot \frac{\partial}{\partial s_2}(r.\sigma_p) - \cos\beta_1 \cdot \frac{\partial}{\partial s_2}(r.\tau_p) = \cos\beta_2 \left(\sigma_t . \sin\beta_1 + \gamma . r . \cos\beta_1\right) \tag{48}$$

Dividing through by $\cos\beta_2$ and $\cos\beta_1$ respectively, and rearranging:

$$\frac{\partial}{\partial s_1}(r.\tau_p) = \tan\beta_2 \left[\frac{\partial}{\partial s_1}(r.\sigma_p) - \sigma_t . \cos\beta_1\right] - \gamma . r . \cos\beta_1 \tag{49}$$

$$\frac{\partial}{\partial s_2}(r.\tau_p) = \tan\beta_1 \left[\frac{\partial}{\partial s_2}(r.\tau_p) - \sigma_t . \cos\beta_2\right] - \gamma . r . \cos\beta_2 \tag{50}$$

In finite difference form:

$$d(r.\tau_p)_1 = \tan\beta_2 \left[d(r.\sigma_p)_1 - \sigma_t (ds_1 . \cos\beta_1)\right] - (\gamma . r).(ds_1 . \cos\beta_1) \tag{51}$$

$$d(r.\tau_p)_2 = \tan\beta_1 \left[d(r.\sigma_p)_2 - \sigma_t (ds_2 . \cos\beta_2)\right] - (\gamma . r).(ds_2 . \cos\beta_2) \tag{52}$$

The pole trail method combines the relationships between the quantities $(r.\sigma_m)$ and $\psi$, and the lengths of the slip line elements $ds_1$ and $ds_2$ in their repsective directions $\beta_1$ and $\beta_2$, according to Eqs. (36) and (37), with the quantities $(r.\tau_p)$ and $(r.\sigma_p)$ with respect to the same elements, according to Eqs. (49) and (50).

This combination of the geometrical and physical relationships can be used to advantage in two-dimensional, plane strain problems to construct the slip line fields geometrically by finite differences.

The method was applied to the problem under consideration but failed to give satisfactory solutions, because of the arbitrariness of the choice between the

terms $\sigma_t \cdot \sin\beta$ and $\gamma \cdot z \cdot \cos\beta$ on the right hand side of Eqs. (36) and (37) and of Eqs. (49) and (50).

The attempt showed, that while the grahpical solution is theoretically valid, it is impractical for solutions involving radial distributions of stresses in three-dimensional problems.

A numerical finite difference method was therefore indicated.

## C. Numerical finite difference method

The differential equations of limit equilibrium were given in (36) and (37).

Assuming that point 1 and point 2 in Fig. 11 are points of the slip line field with known quantities:

$$z_1,\ \sigma_{m_1},\ \psi_1,\ \sin\beta_1,\ \cos\beta_1,\ \sigma_{t_1}$$

and:

$$z_2,\ \sigma_{m_2},\ \psi_2,\ \sin\beta_2,\ \cos\beta_2,\ \sigma_{t_2}$$

Eq. (36) applies to point 1 located on the $s_1$-slip line.

Eq. (37) applies to point 2 located on the $s_2$-slip line.

We provide the relevant quantities with a second suffix to indicate the location of the points.

At point 1 (Eq. 36):

$$\cos\varphi \cdot \frac{\partial}{\partial s_{1,1}}(z_1 \cdot \sigma_{m_1}) + 2(z_1 \cdot \sigma_{m_1}) \sin\varphi \cdot \frac{\partial \psi_1}{\partial s_{1,1}} = -(\sigma_{t_1} \cdot \sin\beta_{2,1} + \gamma z_1 \cdot \cos\beta_{2,1}) \tag{53}$$

At point 2 (Eq. (37):

$$\cos\varphi \cdot \frac{\partial}{\partial s_{2,2}}(z_2 \cdot \sigma_{m_2}) - 2(z_2 \cdot \sigma_{m_2}) \cdot \sin\varphi \cdot \frac{\partial \psi_2}{\partial s_{2,2}} = +(\sigma_{t_2} \sin\beta_{1,2} + \gamma \cdot z_2 \cdot \cos\beta_{1,2}) \tag{54}$$

Point 3 is reached along the $s_1$-slip line through point 1 and along the $s_2$-line through point 2. Omitting the second suffix of $\partial s$, because $\partial s_1$ is related to

point 1 and $\partial s_2$ to point 2, Eqs. (53) and (54) are in finite difference form:

$$\cos\varphi\cdot\frac{d(\tau_1\cdot\sigma_{m_1})}{ds_1} + 2(\tau_1\cdot\sigma_{m_1})\cdot\sin\varphi\,\frac{d\psi_1}{ds_1} = -\left(\sigma_{\tau_1}\cdot\sin\beta_{2,1} + \gamma\cdot\tau_1\cdot\cos\beta_{2,1}\right) \tag{55}$$

$$\cos\varphi\cdot\frac{d(\tau_2\cdot\sigma_{m_2})}{ds_2} - 2(\tau_2\cdot\sigma_{m_2})\cdot\sin\varphi\,\frac{d\psi_2}{ds_2} = +\left(\sigma_{\tau_2}\cdot\sin\beta_{1,2} + \gamma\cdot\tau_2\cdot\cos\beta_{1,2}\right) \tag{56}$$

Since:

$$d(\tau_1\cdot\sigma_{m_1}) = \tau_3\cdot\sigma_{m_3} - \tau_1\cdot\sigma_{m_1}$$

$$d(\tau_2\cdot\sigma_{m_2}) = \tau_3\cdot\sigma_{m_3} - \tau_2\cdot\sigma_{m_2}$$

$$d\psi_1 = \psi_3 - \psi_1$$

$$d\psi_2 = \psi_3 - \psi_2$$

Eqs. (55) and (56) can be written in the following form:

$$\cos\varphi(\tau_3\cdot\sigma_{m_3} - \tau_1\cdot\sigma_{m_1}) + 2(\tau_1\cdot\sigma_{m_1})\cdot\sin\varphi\cdot(\psi_3-\psi_1) = -\left[\sigma_{\tau_1}\cdot\sin\beta_{2,1} + \gamma\cdot\tau_1\cdot\cos\beta_{2,1}\right]ds_1 \tag{57}$$

$$\cos\varphi(\tau_3\cdot\sigma_{m_3} - \tau_2\cdot\sigma_{m_2}) - 2(\tau_2\cdot\sigma_{m_2})\cdot\sin\varphi(\psi_3-\psi_2) = +\left[\sigma_{\tau_2}\cdot\sin\beta_{1,2} + \gamma\cdot\tau_2\cdot\cos\beta_{1,2}\right]ds_2 \tag{58}$$

The unknowns in these two equations are $\tau_3$, $\sigma_{m_3}$, $\psi_3$, $ds_1$ and $ds_2$. Three additional equations are needed to solve for the unknowns. In order to obtain these additional relationships the slip lines will be considered as a sequence of circular arc elements of varying curvature.

When the radius of curvature at a point of the $s_1$-slip line is $R_1$, the increase of $\psi$ in the direction of the slip line, $d\psi_1$, corresponds to a slip line distance, $ds_1 = R_1 d\psi_1$. An identical relationship applies to the $s_2$-slip line; $ds_2 = R_2\cdot d\psi_2$.

This property will be used to establish the coordinates of the intersection point 3 of the $s_1$-slip line and $s_2$-slip line through the points 1. and 2. respectively.

Fig. 11 shows how these coordinates can be computed:

$$r_3 = r_1 + \tfrac{1}{2} ds_1 (\cos\beta_{1,1} + \cos\beta_{1,3}) \tag{59}$$

$$r_3 = r_2 + \tfrac{1}{2} ds_2 (\cos\beta_{2,2} + \cos\beta_{2,3}) \tag{60}$$

and:

$$Z_3 = Z_1 + \tfrac{1}{2} ds_1 (\sin\beta_{1,1} + \sin\beta_{1,3}) \tag{61}$$

$$Z_3 = Z_2 + \tfrac{1}{2} ds_2 (\sin\beta_{2,2} + \sin\beta_{2,3}) \tag{62}$$

From: $\beta_1 = \psi + \tfrac{1}{2}(90^\circ - \varphi) = \psi + \delta$

$\beta_2 = \psi - \tfrac{1}{2}(90^\circ - \varphi) = \psi - \delta$

the trigonometric expressions in $\beta$ can be replaced by expressions in $\psi$ :

$\beta_{2,1} = \psi_1 - \delta$ ; $\beta_{1,1} = \psi_1 + \delta$ ; $\beta_{2,2} = \psi_2 - \delta$

$\beta_{1,2} = \psi_2 + \delta$ ; $\beta_{1,3} = \psi_3 + \delta$ ; $\beta_{2,3} = \psi_3 - \delta$

By substitutions in Eqs. (57) to (62) inclusive we obtain the six equations:

$$\cos\varphi(r_3 \cdot \sigma_{m_3} - r_1 \cdot \sigma_{m_1}) + 2(r_1 \cdot \sigma_{m_1}) \sin\varphi (\psi_3 - \psi_1) = -\left[\sigma_{t_1} \cdot \sin(\psi_1 - \delta) + \gamma r_1 \cdot \cos(\psi_1 - \delta)\right] ds_1 \tag{63}$$

$$\cos\varphi(r_3 \cdot \sigma_{m_3} - r_2 \cdot \sigma_{m_2}) - 2(r_2 \cdot \sigma_{m_2}) \cdot \sin\varphi (\psi_3 - \psi_2) = +\left[\sigma_{t_2} \cdot \sin(\psi_2 + \delta) + \gamma r_2 \cdot \cos(\psi_2 + \delta)\right] ds_2 \tag{64}$$

$$r_3 = r_1 + \tfrac{1}{2} ds_1 \left[\cos(\psi_1 + \delta) + \cos(\psi_3 + \delta)\right] \tag{65}$$

$$r_3 = r_2 + \tfrac{1}{2} ds_2 \left[\cos(\psi_2 - \delta) + \cos(\psi_3 - \delta)\right] \tag{66}$$

$$Z_3 = z_1 + \tfrac{1}{2}ds_1\left[\sin(\psi_1+\delta) + \sin(\psi_3+\delta)\right] \tag{67}$$

$$Z_3 = z_2 + \tfrac{1}{2}ds_2\left[\sin(\psi_2-\delta) + \sin(\psi_3-\delta)\right] \tag{68}$$

with the six unknown quantities: $r_3$, $z_3$, $\sigma_{m3}$, $\psi_3$, $ds_1$ and $ds_2$.

We can therefore solve for the unknowns. The inclusion, however, of the trigonometric functions of $\psi_3$ requires a solution by numerical computations.

<u>The choice of the intermediate principal stress $\sigma_t$</u>

In Chapter II. A we discussed the indeterminedness of the tangential stress $\sigma_t$. In Eq. (28) the factor $\alpha$ expresses this fact.

The value of this factor can be determined *only* at the axis of symmetry and at the boundary of the cavity, where the quantities $r$, $\sigma_m$, $\psi$, $\beta_1$, and $\beta_2$ are known.

Along the axis of symmetry the following quantities are known:

$$r = 0$$

$$\psi = 90^\circ$$

$$\beta_1 = 90^\circ + \left(45^\circ - \frac{\psi}{2}\right) = (90^\circ + \delta)$$

$$\beta_2 = 90^\circ - \left(45^\circ - \frac{\psi}{2}\right) = (90^\circ - \delta)$$

Rewriting Eqs. (36) and (37) in terms of the directional derivatives:

$$\cos\psi\left(\sigma_m \cdot \frac{\partial z}{\partial s_1} + r \cdot \frac{\partial \sigma_m}{\partial s_1}\right) + 2(r \cdot \sigma_m) \cdot \sin\psi \cdot \frac{\partial \psi}{\partial s_1} = -\left[(1 + \alpha \cdot \sin\psi)\sigma_m \cdot \sin\beta_2 + \gamma \cdot r \cdot \cos\beta_2\right]$$

$$\cos\psi\left(\sigma_m \cdot \frac{\partial z}{\partial s_2} + r \cdot \frac{\partial \sigma_m}{\partial s_2}\right) + 2(r \cdot \sigma_m) \cdot \sin\psi \cdot \frac{\partial \psi}{\partial s_2} = +\left[(1 + \alpha \cdot \sin\psi) \cdot \sigma_m \cdot \sin\beta_1 + \gamma \cdot r \cdot \cos\beta_1\right]$$

Introduction of the known quantities of $r, \varphi, \beta_1$ and $\beta_2$, yields:

$$\cos\varphi \cdot \sigma_m \cdot \frac{\partial z}{\partial s_1} = -(1+\alpha \cdot \sin\varphi) \cdot \sigma_m \cdot \sin\beta_2$$

and: $$\cos\varphi \cdot \sigma_m \cdot \frac{\partial z}{\partial s_2} = +(1+\alpha \cdot \sin\varphi) \cdot \sigma_m \cdot \sin\beta_1$$

Dividing through by $\sigma_m$ and putting $\frac{\partial z}{\partial s_1} = \cos\beta_1$ and $\frac{\partial z}{\partial s_2} = \cos\beta_2$, yields:

$$\cos\varphi \cdot \cos\beta_1 = -(1+\alpha \cdot \sin\varphi) \cdot \sin\beta_2$$
$$\cos\varphi \cdot \cos\beta_2 = +(1+\alpha \cdot \sin\varphi) \cdot \sin\beta_1$$

Since: $$\beta_1 = (90^\circ + \delta)$$
$$\beta_2 = (90^\circ - \delta)$$

and: $$\varphi = (90^\circ - 2\delta)$$

we obtain one equation:

$$\cos(90^\circ - 2\delta) \cdot \sin\delta - \cos\delta = \alpha \cdot \sin(90^\circ - 2\delta) \cdot \cos\delta$$

from which:

$$\alpha = -1 \tag{69}$$

hence:

$$\sigma_t = (1 - \sin\varphi)\,\sigma_m \tag{70}$$

The condition of failure and the equilibrium conditions along the axis of symmetry require an intermediate principal stress of the same magnitude as the minor principal stress along that aixs.

The numerical computations of the failure quantities in the stress field were carried out on the assumption that $\alpha = -1$ throughout the medium.

Experimental evidence is required to prove the validity of this assumption. Numerical computations with values of $-1 < \alpha < +1$ will show the significance of variations in the intermediate principal stress.

Chapter III

Conclusions concerning the feasibility of mass earth movement by cavity pressure.

The investigation has shown the feasibility of mass earth movement by initiating a failure condition through a rapid build-up of liquid pressure inside a cavity at some distance from the free boundary of a soil medium.

The magnitude of the critical pressure level depends primarily on the distance of the centre of pressure from the boundary and on the mechanical properties of the soil. The larger the cavity, the smaller the pressure required to cause failure.

In a dry granular material (dry sand) water will penetrate into the soil by the pressure gradient and saturate the immediate vicinity of the cavity. The saturated shell constitutes the intermediary for the transfer of the cavity pressure to the surrounding soil.

In a saturated granular material a rapid build-up of the liquid pressure inside the cavity propagates through the pore-water and reaches the watertable without noticeable delay. As a consequence a critical flow gradient inside the soil medium develops and while the entire soil mass becomes involved in a failure condition, erosion of the cohesionless material is initiated at the boundary.

In a saturated or dry cohesive material the propagation of the cavity pressure depends on the nature of the clay material. When the void channel system constitutes a "closed" system, the cavity pressure will be transferred mostly to the immediate vicinity of the cavity and failure will occur as if the material would be an impervious medium.

In materials with an "open" void channel system the cavity pressure creates a condition comparable to that of a saturated granular medium.

In all soil media the state of failure by rapid pressure build-up is followed by deterioration of the material. The time period for maintaining critically high liquid pressures has shown to be comparitively short(in the order of seconds). The volume displacements as a result of the expansion of the cavity, preceding the state of failure, are comparitively small. In the first phase of the treatment high pressure input and a small water quantity are required. In the second phase the pressure can be decreased, but a larger quantity of water is required to maintain a liquid flow of soil particles and deteriorated soil materials.

Example of a continuously operating mass earth movement system

As an example of a continuously operating system, Fig. 11 shows a diagram of the mechanism required to move the soil in excavating a canal profile.

A liquid pressure feeding line is located across the canal axis and connected with a number of casings, which are gradually lowered into the soil, while pressure is constantly applied. The liquid flow moves the soil out and a system of suction lines at the toe line removes the suspended material.

This material can be pumped over the sides and deposited alongside at any suitable location.

The excavation is supposed to remain in open connection with a reservoir (river, ocean) allowing for a constant, unlimited supply of water.

Water is used both as a pressure medium and a transportation device. The entire mechanism can be moved along the canal trace, while the liquid material is deposited alongside.

By adding suitable chemical admixtures the deterioration of cohesive material can be speeded up. When pseudo-rock materials are encountered the initiation of a failure condition, using liquid pressure, can be promoted by using suitable explosives inside the cavity. In granular materials and cohesive soils of medium strength such measures will not be necessary, provided that an average depth of the cavity is maintained.

In cohesionless, dry granular materials the addition of water to the soil promotes the occurrence of flow slides, which-when kept in continuous motion by transportation through the suction lines-do require little energy.

In cohesionless, wet granular material the removal of material along the toe of the slope will generally be sufficient to cause flow slides as a continuous earth moving mechanism.

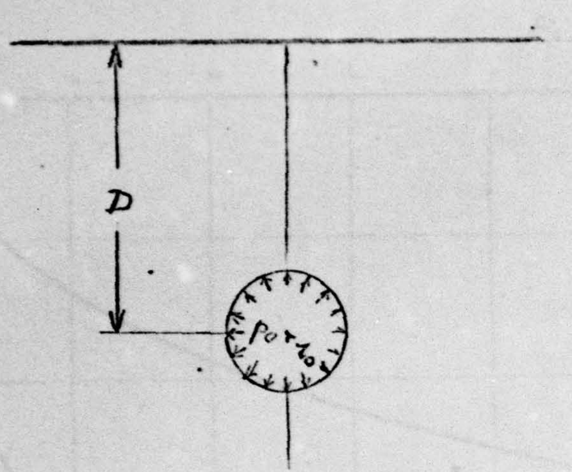



FIG. 1

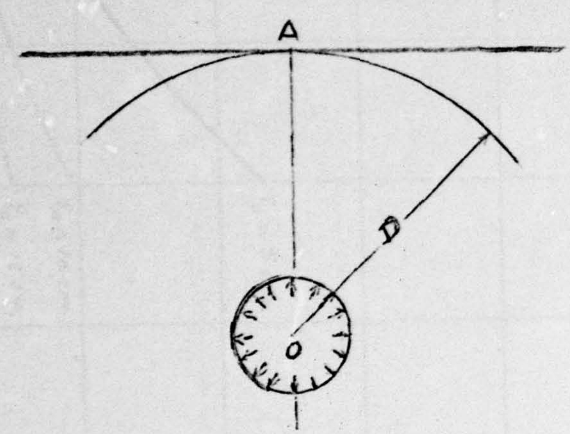



FIG. 2

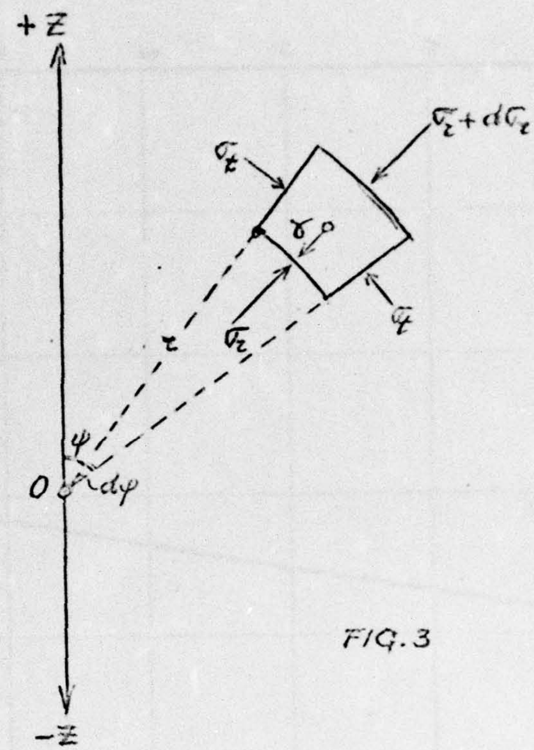



FIG. 3

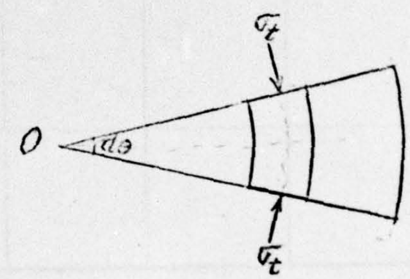
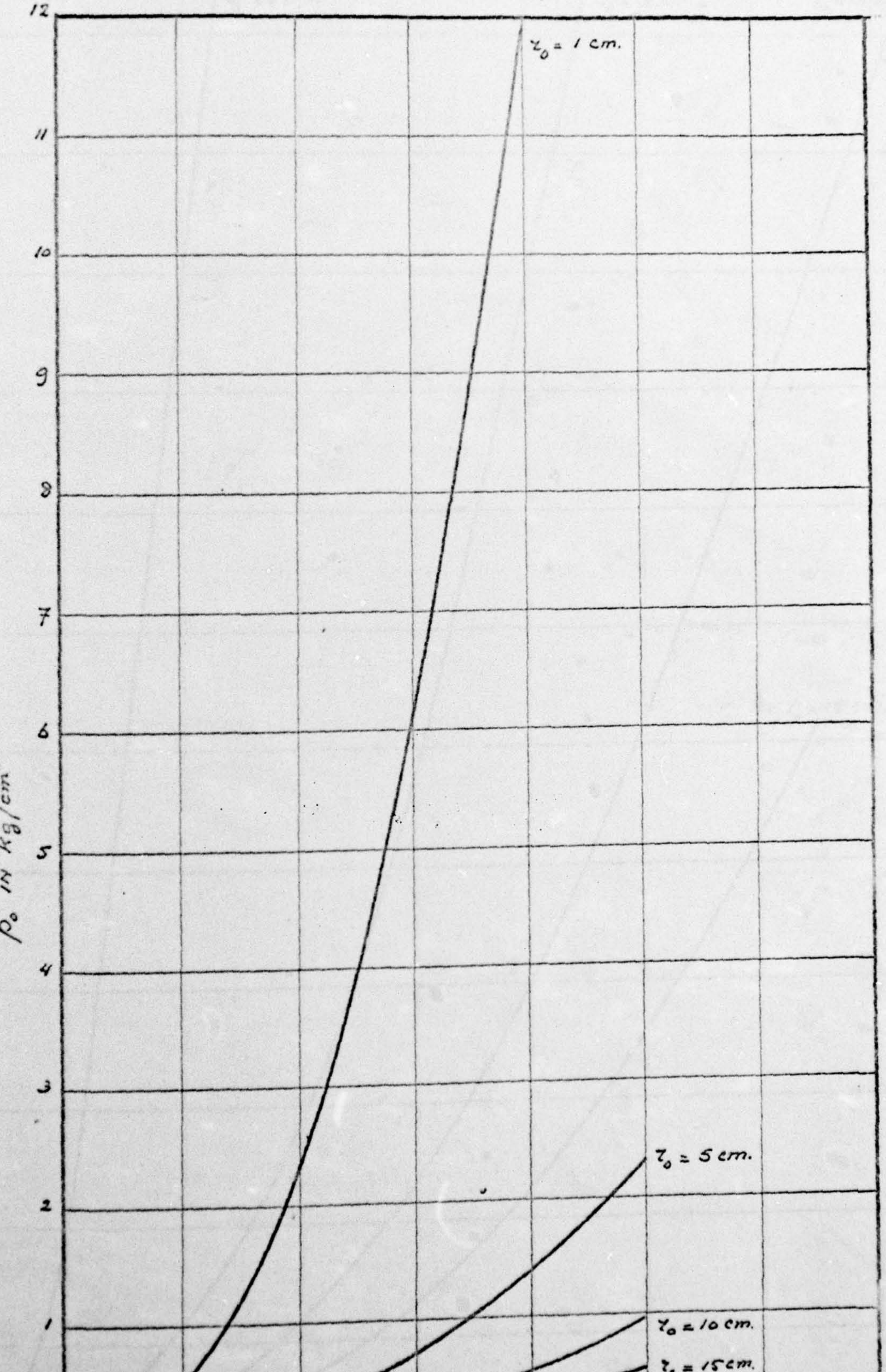
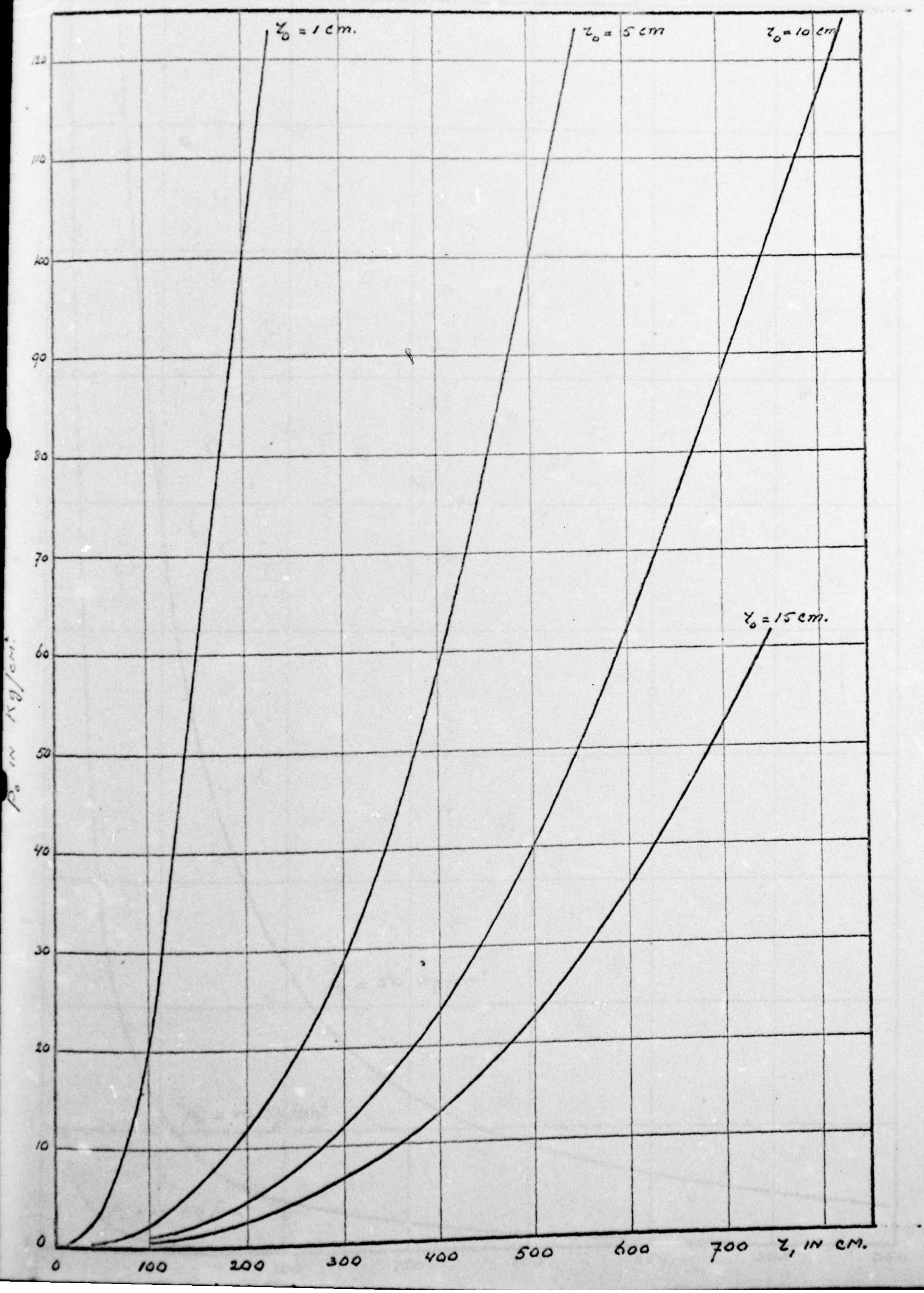


FIG. 4



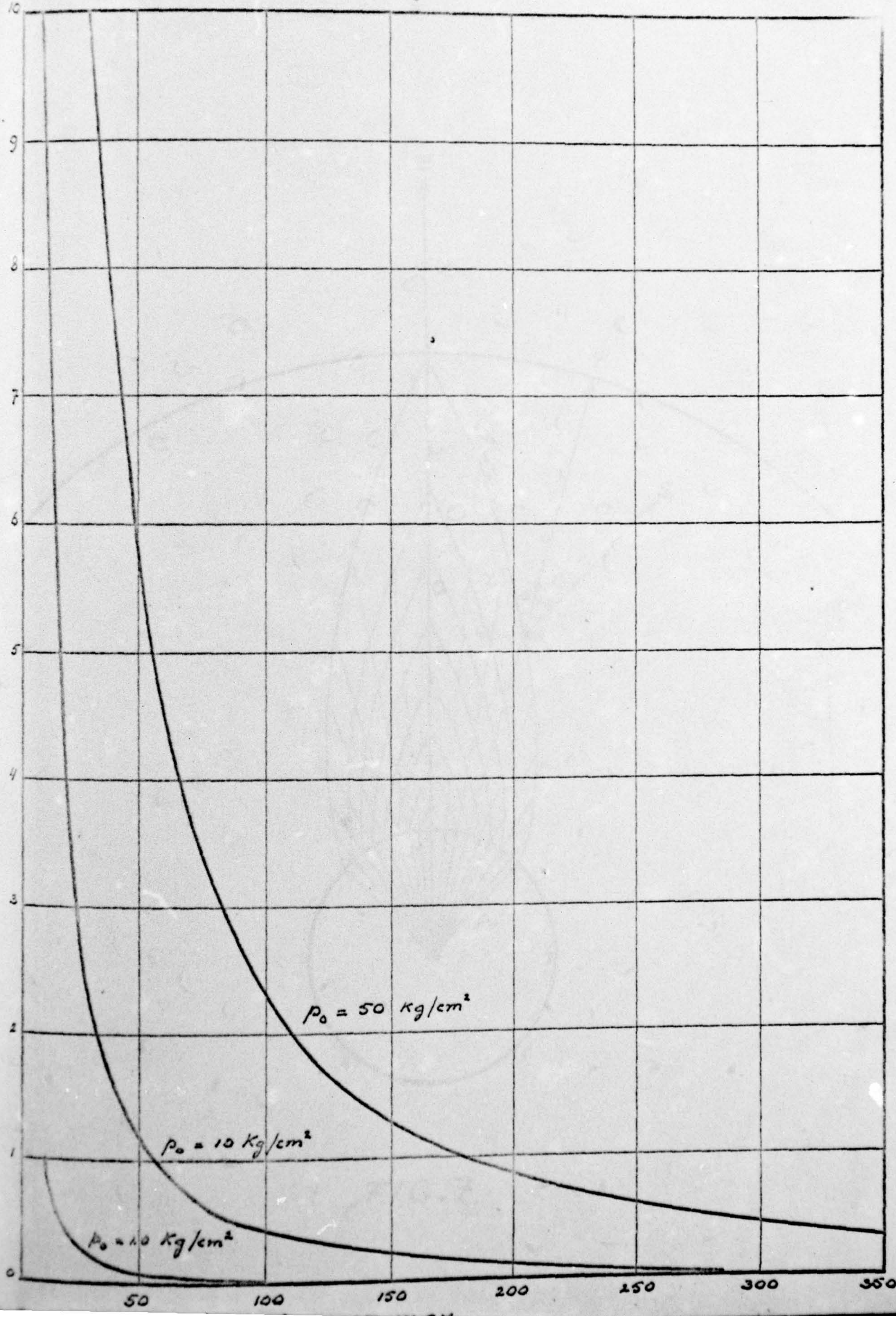
12
11
10
9
8
7
6
5
4
3
2
1
0
$p_o$ IN Kg/cm²
$r_o$ = 1 cm.
$r_o$ = 5 cm.
$r_o$ = 10 cm.
$r_o$ = 15 cm.
0
20
40
60
80
100
$r_1$ IN CM.

FIG. 5



$r_0 = 1$ cm.
$r_0 = 5$ cm
$r_0 = 10$ cm
$r_0 = 15$ cm.
$p_0$ IN Kg/cm.²
$r$, IN CM.
0
10
20
30
40
50
60
70
80
90
100
110
120
0
100
200
300
400
500
600
700

FIG. 6



10
9
8
7
6
5
4
3
2
1
0
50
100
150
200
250
300
350
$P_0 = 50$ Kg/cm$^2$
$P_0 = 10$ Kg/cm$^2$
$P_0 = 1.0$ Kg/cm$^2$

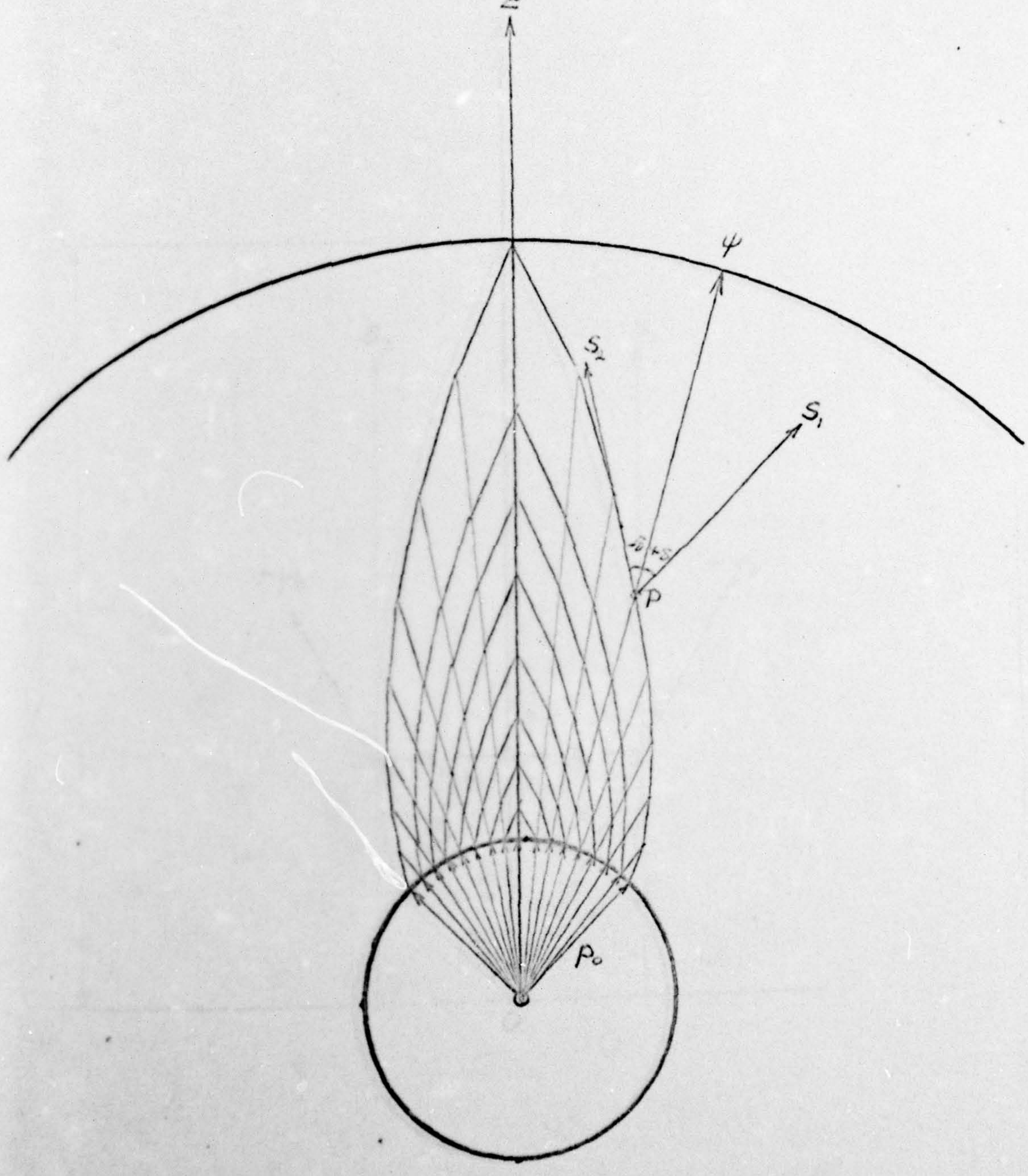



FIG. 7

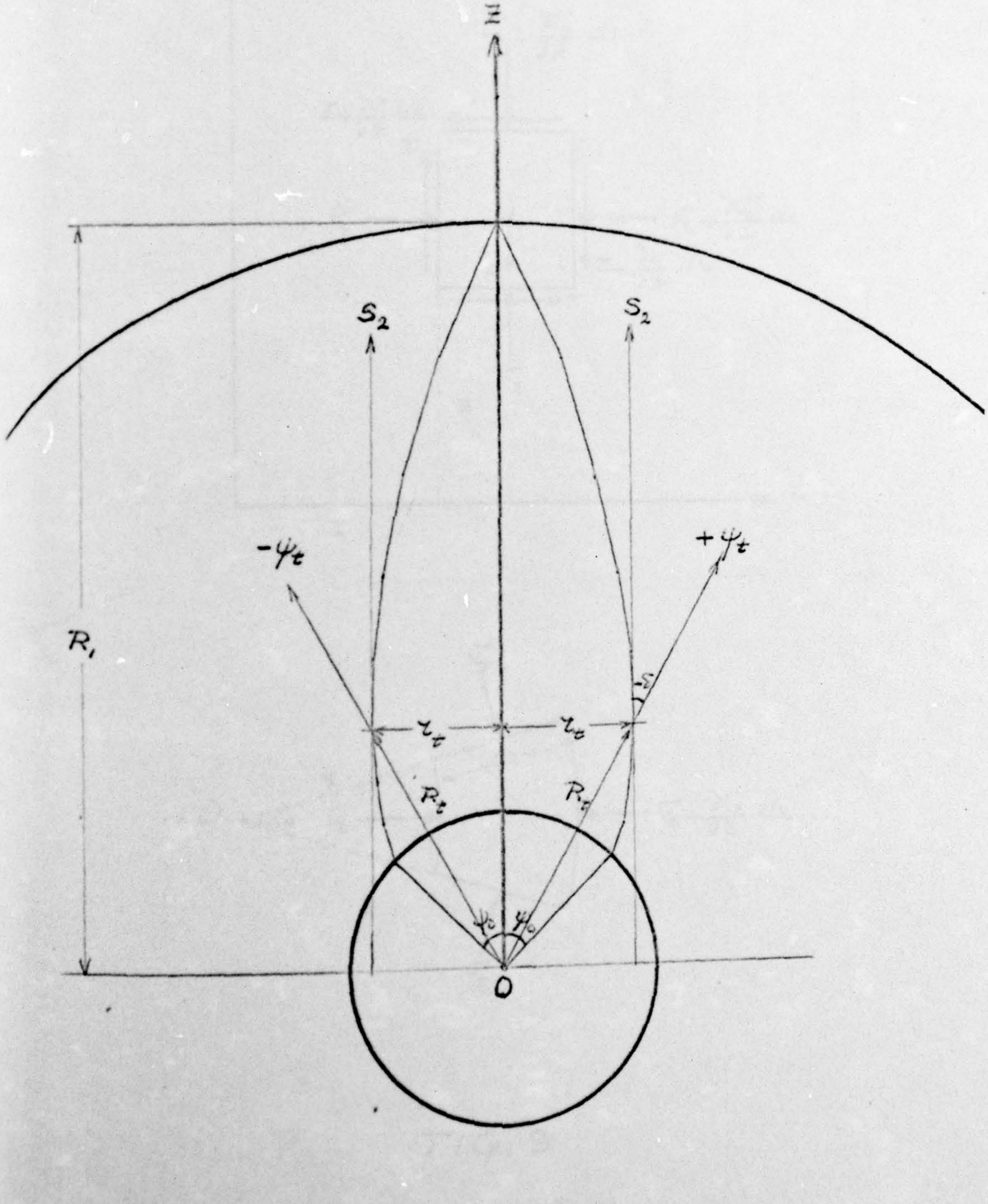



FIG. 8

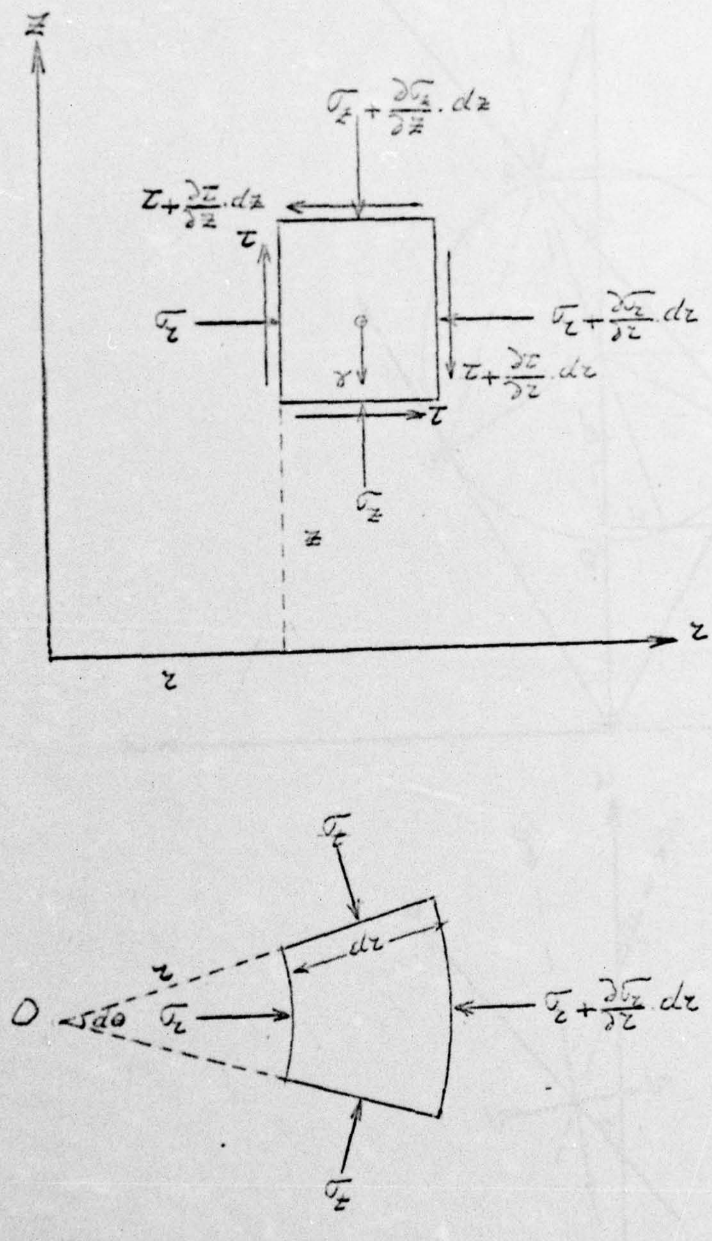
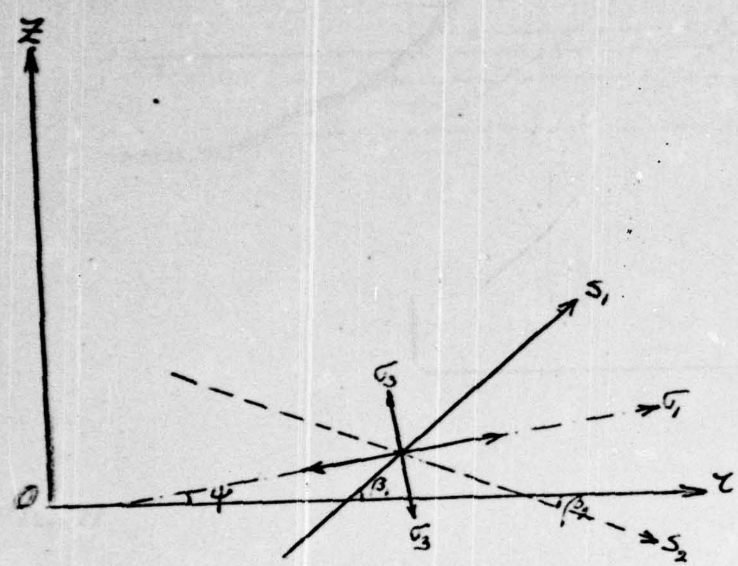

z
σz + ∂σz/∂z . dz
τ + ∂τ/∂z . dz
τ
σr
σr + ∂σr/∂r . dr
τ + ∂τ/∂r . dr
γ
τ
σz
z
r
r
σt
dr
r
O
dθ
σr
σr + ∂σr/∂r . dr
σt

FIG. 9

PHYSICAL PLANE




STRESS PLANE

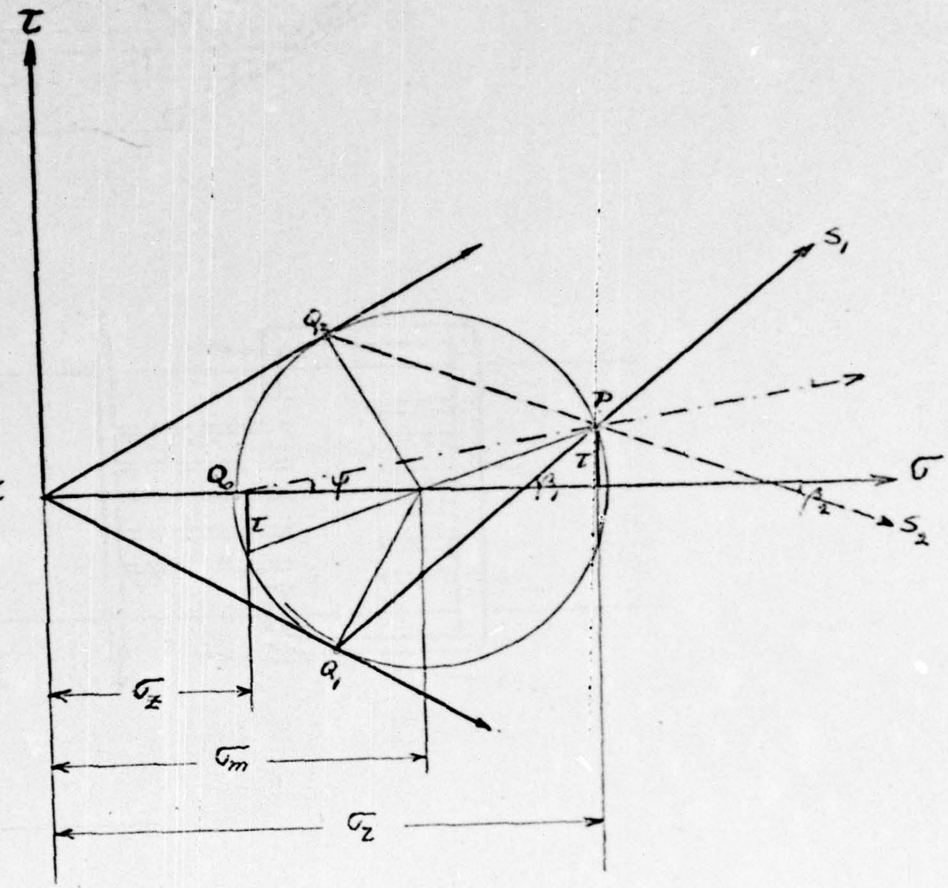



FIG. 10

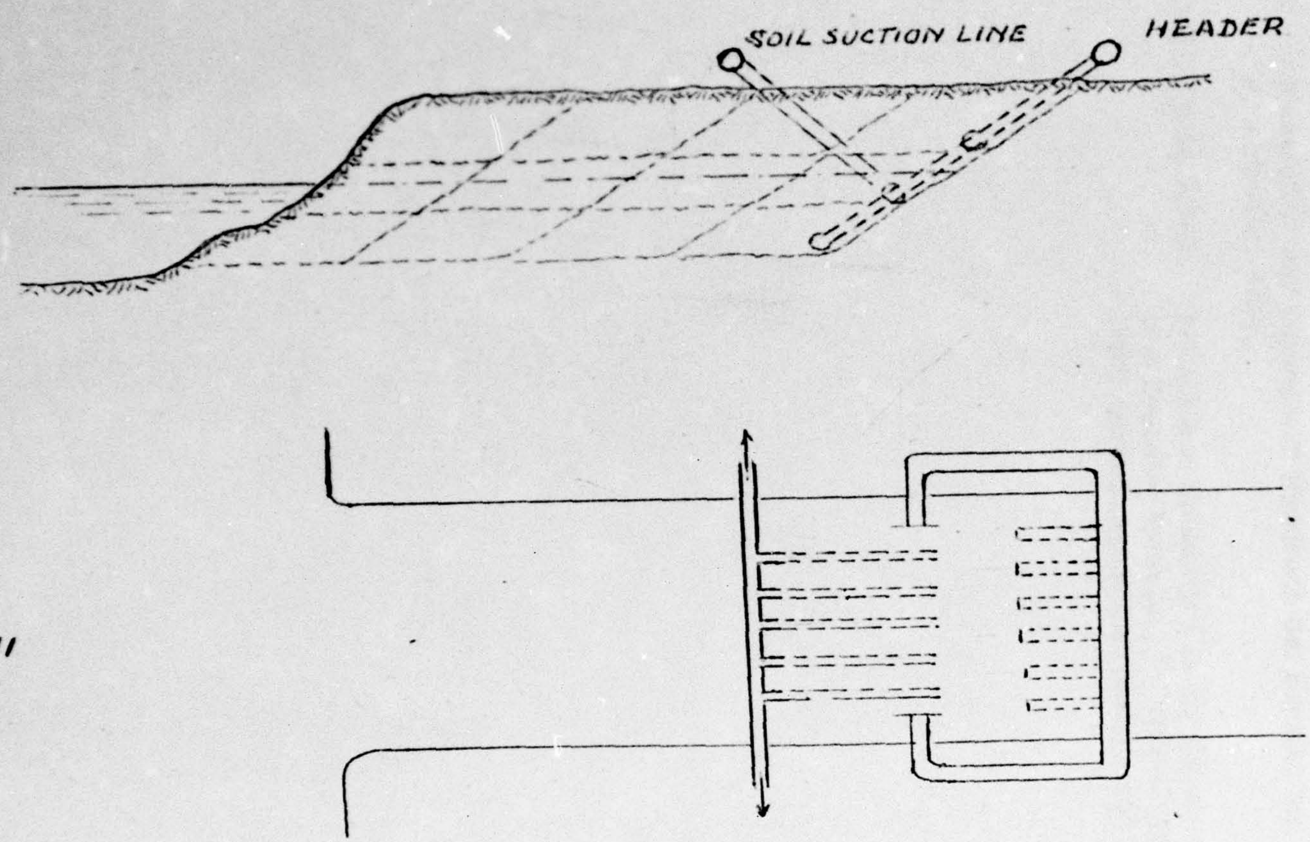



FIG. 11

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